CHAPTER 8

Nonlinearities

A process is said to be *linear* if the process response is proportional to the stimulus given to it. For example, if you double the amount deposited in a conventional savings account (the stimulus), then you will receive double the interest (the response). Similarly, if you work ten percent longer hours, you would hope to accomplish ten percent more work. These are linear responses.

Models that assume a process is linear have been extensively studied because the mathematics for such models is relatively straightforward, and linear models can adequately represent the behavior of many realistic processes over a useful range of conditions. It is often possible to solve the equations for linear models without the need to use computers. Thus, in the era before the widespread availability of computers, the ease of solution for linear models led to their use even in situation where the real-world process was known to be *nonlinear*.

Many business processes are nonlinear, especially when pressed to extremes. For example, while it may be true that if you work ten percent longer hours you will accomplish ten percent more work, it is probably not true that if you work twice as many hours you will accomplish twice as much work. Many of us have attempted to do this, and have soon suffered from burnout leading to a reduction in our working effectiveness. This is a nonlinear response. Similarly, the available production capacity may limit the amount of a product that can be sold, regardless of the amount of sales effort or the degree of customer demand.

In other cases, such as graduated income taxes or variable interest rates on money market accounts, nonlinear responses are deliberately designed into the system. With graduated income taxes, the amount of tax grows more rapidly than the increase in income, and with a money market account the rate of interest may grow more than proportionally as the balance grows.

The simulation approach presented in preceding chapters can be extended to address nonlinear effects without much difficulty. This capability for readily modeling nonlinear processes is an advantage of simulation over hand calculation methods. With hand calculation, nonlinear situations can be complex to address. With simulation, it is often as straightforward to model nonlinear situations as to model ones that are linear.

8.1 Nonlinear Responses

Figure 8.1 shows the stock and flow diagram, Vensim equations, and a graph of Savings Balance for a conventional savings account with compound interest where the interest is left to accumulate in the account for 20 years. The interest rate is five percent (0.05) per year, and the initial balance is $900. After 20 years, the balance has grown to a little over $2,400. The response (that is, the earned interest) is linearly related to the initial amount placed in the account.

Using IF THEN ELSE to Model Nonlinear Responses

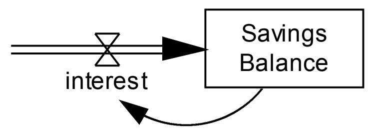
Some money market accounts have a sliding interest rate where the interest rate depends on the balance in the account. For example, suppose that an interest rate of five percent (0.05) per year is paid on every dollar in the account up to $1,000, and an interest rate of ten percent (0.10) per year is paid on every dollar in the account over $1,000. Then the interest is given by

$$\text{interest} = \begin{cases} 0\;05 \times \text{Savings Balance} & \text{Savings Balance} < \$1\;000 \\ 0\;05 \times 1\;000 & \\ \quad +0\;10 \times (\text{Savings Balance} - 1\;000) & \text{otherwise} \end{cases}$$

A somewhat generalized version of this model is shown in Figure 8.2. In Figure 8.2, the Savings Balance amount at which the interest rate changes is specified by the constant BREAKPOINT (which is 1,000 for this example), the interest rate paid on each dollar below BREAKPOINT is specified by the constant LOW RATE (which is 0.05), and the interest rate paid on each dollar above BREAKPOINT is specified by the constant HIGH RATE (which is 0.10). (The use of these constants, rather than hard wiring in specific values for BREAKPOINT, LOW RATE, and HIGH RATE, facilitates sensitivity analysis using the automated features of Vensim. This is discussed further below.)

From Figure 8.2c, we see that the Savings Balance after 20 years is over $3,400, which is substantially more than with the conventional savings account shown in Figure 8.1. (Note that the increase in savings rate for most real-world money market savings accounts above the BREAKPOINT is usually not as great as shown in this example! The large value used in this example for HIGH RATE makes it easier to see the impact of the nonlinear interest rate.) A detailed examination of the model output shows that the Savings Balance exceeds the BREAKPOINT value of $1,000 during the second year, and after that the money market account generates more interest than the conventional savings account.

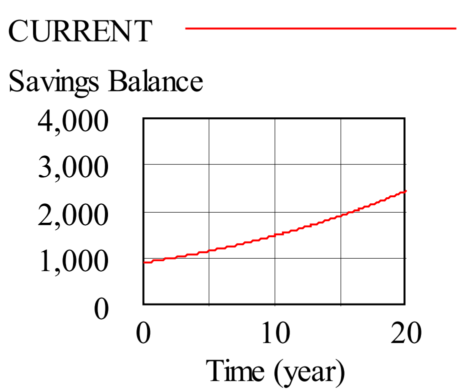
It is straightforward to modify the model in Figure 8.2 to demonstrate that this process is nonlinear. Specifically, the amount of interest earned by the account is not linearly proportional to the initial Savings Balance. As an example, you may wish to verify that when the initial Savings Balance is doubled to $1,800, the final Savings Balance after twenty years almost triples from about $3,500 to almost $10,000. This happens because the modified initial balance of $1,800 is greater than $1,000. Therefore, each dollar of interest earned is compounded at



a. Stock and flow diagram

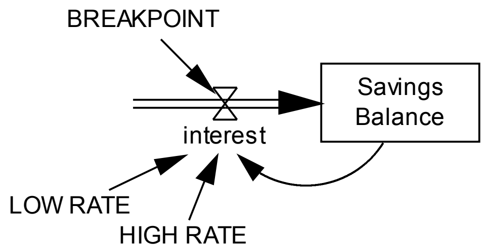
```
(1) FINAL TIME  = 20
(2) INITIAL TIME  = 0
(3) interest = 0.05 * Savings Balance
(4) SAVEPER  = TIME STEP
(5) Savings Balance= INTEG (interest, 900)
(6) TIME STEP  = 0.125
```

b. Vensim equations



c. Savings balance

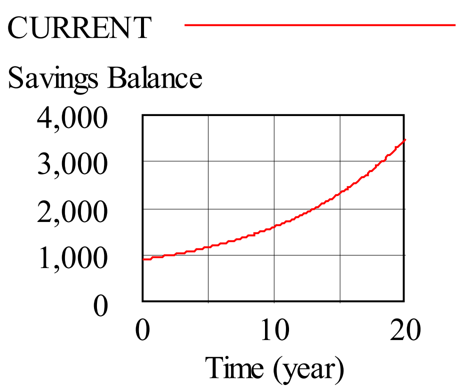
Figure 8.1 *Model for a conventional savings account*



a. Stock and flow diagram

```
(01) BREAKPOINT = 1000
(02) FINAL TIME  = 20
(03) HIGH RATE = 0.1
(04) INITIAL TIME  = 0
(05) interest=
        IF THEN ELSE(Savings Balance < BREAKPOINT,
          LOW RATE * Savings Balance,
          LOW RATE * BREAKPOINT
            + HIGH RATE * (Savings Balance - BREAKPOINT))
(06) LOW RATE = 0.05
(07) SAVEPER  = TIME STEP
(08) Savings Balance= INTEG (interest, 900)
(09) TIME STEP  = 0.125
```

b. Vensim equations



c. Savings balance

Figure 8.2 *Model for a money market savings account (IF THEN ELSE)*

HIGH RATE from the beginning, while this does not happen with the interest for the Initial Balance of $900 specified in Figure 8.2 until the Savings Balance reaches $1,000.

The IF THEN ELSE feature illustrated in equation (05) of Figure 8.2b provides a powerful and flexible way to model this type of nonlinear response. For example, it is possible to nest a second IF THEN ELSE within the first one to handle a situation where there is a second breakpoint at which the interest rate earned on each dollar changes again.

Using Lookup Functions to Model Nonlinear Responses

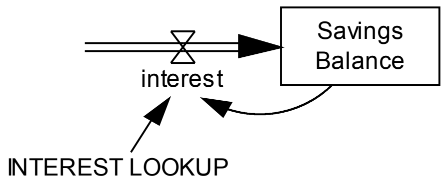
In addition to the IF THEN ELSE function, another approach to modeling nonlinear responses is provided by many simulation languages using lookup functions. A Vensim model for the money market account example which uses a lookup function is shown in Figure 8.3. With this approach, the nonlinear response function (which is interest for this example) is modeled by entering several pairs of points. The simulation program then creates a curve through these points which is used to determine the necessary values to run the simulation.

Equation (4) of Figure 8.3b defines this lookup function, which is called INTEREST LOOKUP. This function is specified by the three pairs of points (0, 0), (1000, 50), and (2000, 150). These points specify that there is $0 of interest per year earned on a Savings Balance of $0, $50 of interest earned per year on a Savings Balance of $1,000, and $150 of interest earned per year on a Savings Balance of $2,000. In Vensim, the lookup function calculates intermediate values by drawing straight lines between the specified pairs of values. Thus, the complete lookup function is shown in Figure 8.3c.

A casual examination of the Figure 8.2 and Figure 8.3 models indicates that these are the same, and thus they should show the same Savings Balance. However, a comparison of Figure 8.2c with Figure 8.3d shows that the Savings Balance curves are somewhat different. What has happened?

The difference between the curves shown in Figure 8.2 and Figure 8.3 illustrates a potential difficulty with using lookup functions. The lookup function in equation (4) of Figure 8.3b is specified over a range of values for Savings Balance from $0 to $2,000. However, the actual Savings Balance exceeds $2,000 during the thirteenth year. The specified behavior for a lookup function in Vensim when the range is exceeded is to clamp the output of the function at the highest specified value. Thus, whenever the Savings Balance is above $2,000, the lookup function INTEREST LOOKUP gives an output of $150. Clearly, this is incorrect for this savings account! (Vensim generates a warning message whenever the range specified for a lookup function is exceeded. In this particular case the following message is generated: `WARNING: At 13,25 Above INTEREST LOOKUP computing interest.`)

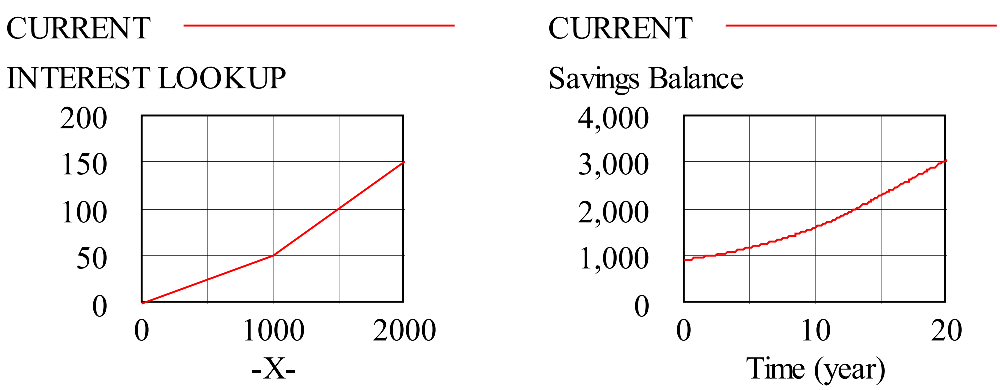
To correct this problem, it is necessary to widen the range over which INTEREST LOOKUP is specified. This can be done by replacing equation (4) in Figure 8.3b with the following:



a. Stock and flow diagram

```
(1) FINAL TIME  = 20
(2) INITIAL TIME  = 0
(3) interest = INTEREST LOOKUP(Savings Balance)
(4) INTEREST LOOKUP([(0,0)-(2000,200)],(0,0),(1000,50),(2000,150))
(5) SAVEPER  = TIME STEP
(6) Savings Balance= INTEG (interest, 900)
(7) TIME STEP  = 0.125
```

b. Vensim equations



c. Lookup function

d. Savings balance

Figure 8.3 *Model for a money market savings account (lookup function)*

```
(4) INTEREST LOOKUP([(0,0)-(5000,500)],(0,0), (1000,50),(5000,450))
```

which expands the upper limit for the range of INTEREST LOOKUP from a Savings Balance of $3,000 up to $5,000. When this change is made, identical output is generated to that shown in Figure 8.2c.

Comparison of IF THEN ELSE and Lookup Functions

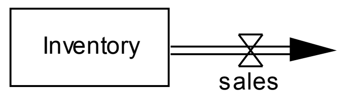
The IF THEN ELSE and lookup functions each have advantages and disadvantages for modeling nonlinear functions. As Figure 8.2b shows, it is possible to include constants in an IF THEN ELSE function so that a sensitivity analysis can be conducted directly in terms of the model constants BREAKPOINT, LOW RATE, and HIGH RATE using the automated procedures in Vensim. This cannot be done so directly when a lookup function is used. (Vensim does support a sensitivity analysis feature where a lookup function can be temporarily changed for a particular model run, but some calculation is necessary to determine exactly how the lookup function points must be changed to represent particular low and high interest rates for the money market account model.)

On the other hand, a lookup function can easily be constructed for situations where there are more than one breakpoint. While this can be done with IF THEN ELSE functions by nesting them, this can lead to complex function expressions.

8.2 Resource Constraints

Another common cause of nonlinear responses in a business process is resource constraints, such as limits on available personnel or production capacity. Figure 8.4 illustrates a first attempt at a model for a simple situation of this type where there is a fixed Inventory of 100,000 units available to sell, and a sales rate of 10,000 units per week. As the curves in Figure 8.4c demonstrate, this first model is inadequate. The Inventory drops to zero at ten weeks, but sales continue undiminished at a rate of 10,000 units per week. While an order backlog might grow for a while when the product is not available, sales are likely to drop as customers cannot get the product. The constraint on the number of units available to sell needs to be included in the model.

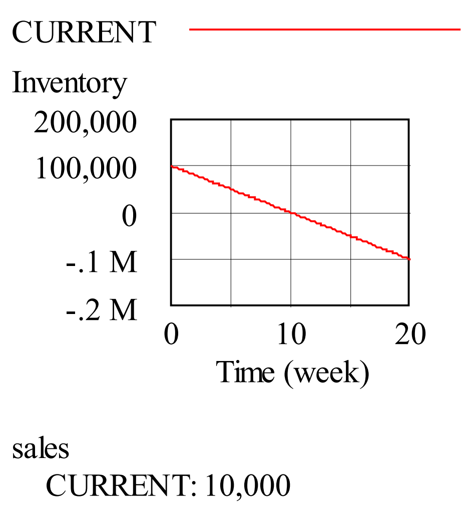
Figure 8.5 shows a modification to the Figure 8.4 model that uses an IF THEN ELSE function to shut off sales when Inventory reaches zero.



a. Stock and flow diagram

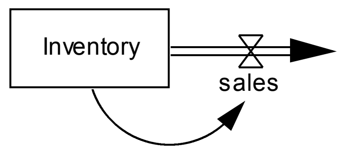
```
(1) FINAL TIME  = 20
(2) INITIAL TIME  = 0
(3) Inventory = INTEG (-sales, 100000)
(4) sales = 10000
(5) SAVEPER  = TIME STEP
(6) TIME STEP  = 0.125
```

b. Vensim equations



c. Inventory and sales

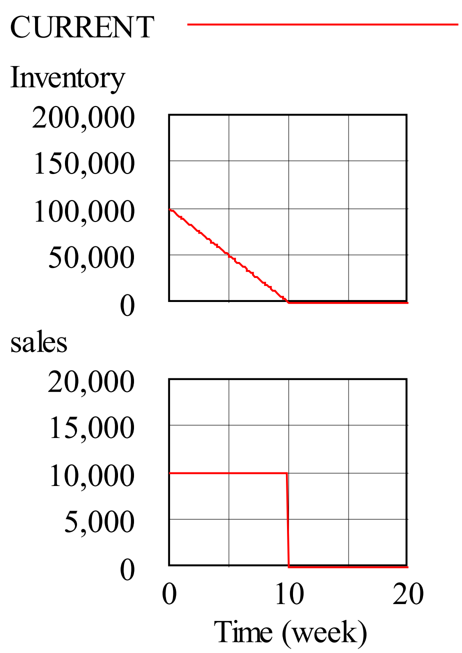
Figure 8.4 *Initial model for sales*



a. Stock and flow diagram

```
(1) FINAL TIME  = 20
(2) INITIAL TIME  = 0
(3) Inventory = INTEG (-sales, 100000)
(4) sales = IF THEN ELSE(Inventory > 0, 10000, 0)
(5) SAVEPER  = TIME STEP
(6) TIME STEP  = 0.125
```

b. Vensim equations



c. Inventory and sales

Figure 8.5 *Modified model for sales*